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LETTER TO THE EDITOR

Ising model on a 3D Sierpinski gasket with a non-trivial phase transition

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Abstract. We derive and solve exact recursion relations for a multispin interaction Ising model on a 3D Sierpinski gasket. In addition to the high- and low-temperature fixed points we find three non-trivial fixed points, which govern the behaviour of two new, 'multiple-of-2', phases. These phases are characterised by the absence of any tetrahedron with an odd number of up spins.

There has recently been an upsurge of interest in solving statistical mechanics problems formulated on fractal lattices (e.g. Gefen *et al* 1981, 1983, 1984a, b, Alexander 1981, Stephen 1981). The reason for this is at least twofold. A regular fractal can be considered as a crude approximation for a disordered system at its percolation threshold, and results obtained for fractals may have some relevance for real random systems. More important, however, is that exact renormalisation group (RG) transformations can be devised for a certain class of so-called finitely ramified fractals (Nelson and Fisher 1975, Gefen *et al* 1984a). A fractal with finite ramification can be cut into two disjoint pieces by removing a finite number of bonds. Unfortunately, the Ising model with only nearest-neighbour interactions does not seem to have any phase transition at non-zero temperature on such a lattice. The purpose of this letter is to show how introduction of multisite interactions may result in a non-trivial phase transition.

Consider a three-dimensional Sierpinski gasket. A topologically equivalent construction is shown in figure 1, but the reader who wishes to see a more artistic view

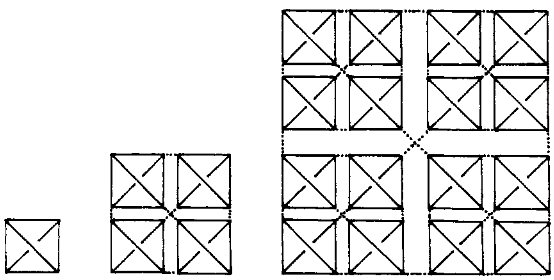

Figure 1. Stagewise build-up of a topological equivalent to the 3D Sierpinski gasket. The endpoints of the dotted lines are to be identified, or equivalently, the broken lines carry infinite interactions. The unit cell is topologically a tetrahedron.

should consult Mandelbrot's book (1982, p 143). We take the negative of the Hamiltonian to be

$$-\mathscr{H}=\sum_{\langle ijkl\rangle}[J_4\delta_4(s_i,s_j,s_k,s_l)+J_3\delta_3(s_i,s_j,s_k,s_l)]+E_0. \tag{1}$$

The sum is performed over all elementary tetrahedra, and we have introduced new functions $\delta_n(\ldots)$. These are defined to equal unity provided that exactly n of its arguments are identical, and zero otherwise. Of course, we could have used ordinary δ functions to express the same relations, but the δ_n's provide a compact and convenient notation. Our model is the same as the one considered by Gefen *et al* (1984a), but we use different variables to describe it. Our coupling constants can be expressed in terms of the interactions appearing in their Hamiltonian (3.6):

$$J_4=8K, \qquad J_3=2K-2L, \qquad E_0=-2K+L+A. \tag{2}$$

In particular, the ordinary nearest-neighbour Ising model is embedded along the line $J_4=4J_3$.

Exact decimation yields the following recursion relations for $x=\exp(J_4)$ and $y=\exp(J_3)$.

$$\begin{aligned} x' &= \frac{3+30y^2+8y^4+x(4+12y^2)+6x^2y^2+x^4}{2+26y^2+8y^4+x(4+20y^2)+x^2(2+2y^2)} \\ y' &= y\cdot\frac{13+24y^2+x(15+8y^2)+3x^2+x^3}{2+26y^2+8y^4+x(4+20y^2)+x^2(2+2y^2)}. \end{aligned} \tag{3}$$

A sketch of the RG flows in parameter space appears in figure 2. There are five different fixed points. The stable high-temperature fixed point is located at $(x,y)=(1,1)$ and the marginally unstable low-temperature fixed point lies at (∞,∞). From the latter fixed point a semi-stable line emerges. The equation of this line is to first order $x=2y^2$ or $J_4=2J_3+\ln 2$. Points outside the line are attracted to it, and then they move very

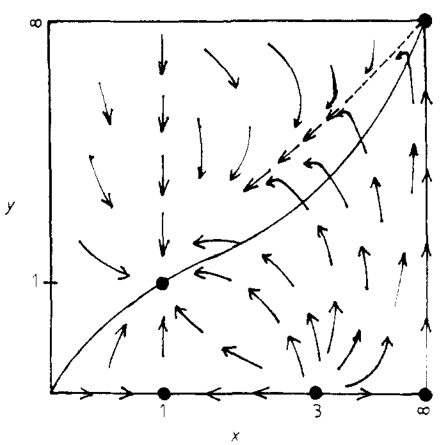

Figure 2. Renormalisation group flows in the xy plane. The full line indicates the location of the nearest-neighbour Ising model. Fixed points are marked with dots. The broken line is the semi-stable line emerging from the low-temperature fixed point.

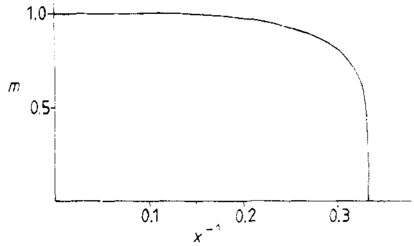

Figure 3. Spontaneous magnetisation m as a function of x^{-1}, when $y=0$.

slowly away from the fixed point along this line. This corresponds to a dramatic increase in the correlation length, and in fact this length diverges as an exponential of an exponential of the inverse temperature.

The phase diagram is not exhausted with the above, however. On the line $y=0$ we find three additional fixed points, at $x=1$, $x=3$, and $x=\infty$. A vanishing y implies that no elementary tetrahedron can have three spins pointing in the same direction, and the fourth in the opposite one. The fixed point $x=1$ acts as a sink for a new disordered phase, which exhibits no long-range magnetisation but excludes all configurations where any tetrahedron has an odd number of up spins. We propose to call this phase a disordered 'multiple-of-2' phase. Similarly, we have an ordered multiple-of-2 phase, which flows into $x=\infty$ under the RG. $x=3$ are the critical fixed points that divide these two regimes. The thermal scaling powers at the critical point are $y_1=1$ in the direction of the y axis, and $y_2=\ln\frac{7}{2}/\ln 2\approx 1.807\,36$ away from it.

In order to calculate the magnetic scaling power we add a symmetry-breaking term to the Hamiltonian.

$$-\mathscr{H}_{\rm sb}=\sum_{\langle ijkl\rangle} h[2\delta_5(s_i, s_j, s_k, s_l, 1)-1]. \tag{4}$$

In the subspace $y=0$, the recursion relation for h becomes to first order

$$h'=\frac{4x+4x^4}{3+4x+x^4}\cdot h\equiv a(x)\cdot h, \tag{5}$$

from which we deduce that the magnetic scaling power $y_h=\ln\frac{7}{2}/\ln 2\approx 1.807\,36$. From (5) we can also calculate the spontaneous magnetisation, at any temperature, using the formula

$$m(x)=\prod_{i=0}^{\infty}\frac{a(x^{(i)})}{b^D}, \tag{6}$$

where $b=D=2$ for the Sierpinski gasket, and $x^{(i)}$ is the ith iterate of x. The spontaneous magnetisation is plotted in figure 3.

It is easy to understand the origin of these new phases. If we let each elementary tetrahedron contain only 0, 2, or 4 up spins, there is no way to arrange four tetrahedra so that the renormalised object (the 'supertetrahedron') has an odd number of up spins. It is non-trivial, however, that this 'multiple-of-2' system exhibits a continuous phase transition at a finite value of the coupling constant.

We can easily generalise the multiple-of-2 phases to Sierpinski gaskets in arbitrary dimension. The unit simplex (triangle, tetrahedron, etc.) has $d+1=p_1\cdot p_2\cdot\ldots\cdot p_N$ corners, where the p_i's are prime numbers. If each unit simplex has a multiple of p_1 up spins, the 'supersimplex' obtained after renormalisation will also have this. Hence, unless $d+1$ is a prime number, we will have different 'multiple-of-p' phases. Whether or not there will be any non-trivial phases transitions among these phases remains of course an open question, but our personal opinion is that it is very likely.

In conclusion, we have solved exactly an Ising model with multispin interactions on the three-dimensional Sierpinski gasket. In addition to the fixed points found by Gefen *et al*, we also found three new fixed points, corresponding to 'multiple-of-2' phases. Although the transition between these does not strictly occur at a non-zero temperature, since $J_3=-\infty$, it is to our knowledge the first example of a non-trivial phase transition in a spin model on a finitely ramified fractal lattice. Unfortunately, the existence of multiple-of-2 phases is possible only on extremely regular structures, like Sierpinski gaskets. On the disordered systems, which the gaskets are supposed to model, this regularity is not present. Consequently, the phase transition which we found in this letter does not have any relevance for real systems, but represents a mere mathematical amusement.

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